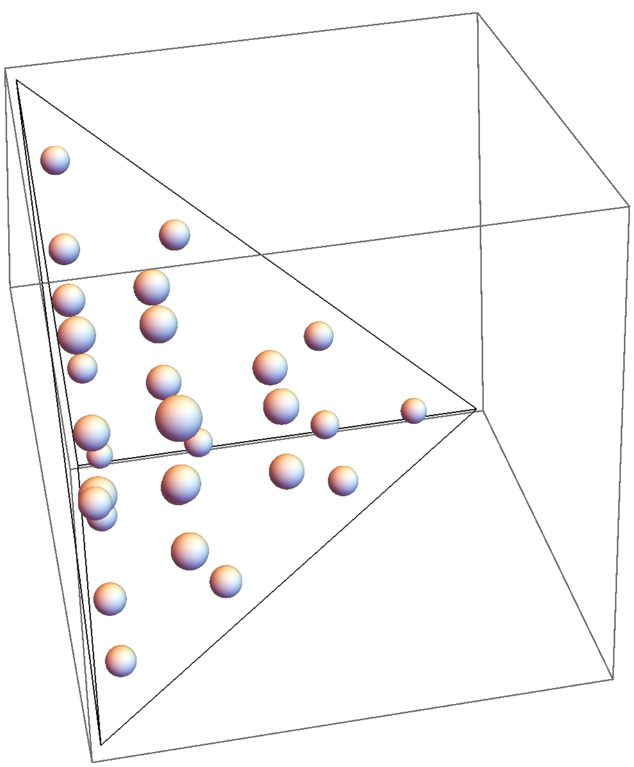
```
In[82]:= Remove["Global`*"]
(* Y.Jinyun,Symmetric gaussian quadrature formulae for tetrahedronal regions,
Comp.Meth.Appl.Mech.Eng.43 (1984),349--353.*)
(* http://nines.cs.kuleuven.be/ecf/mtables.html *)

(* Here we export the integration point coordinates and the weights for *)
(* the reference tetrahedron (0,0,0|1,0,0|0,1,0|0,0,1) *)

(* These are used for the cyclic permutation *)
F1 = {{0, 0, 1}, {0, 1, 0}, {-1, -1, -1}};
F2 = {{0, 0, 1}, {-1, -1, -1}, {1, 0, 0}};
F3 = {{-1, -1, -1}, {0, 0, 1}, {0, 1, 0}};

(* Create empty tables for weights and coordinates *)
NGP = 29;
weights = Table[ToExpression[StringJoin["w", ToString[i]]], {i, 1, NGP}];
coords = Table[{
     ToExpression[StringJoin["x", ToString[i]]],
     ToExpression[StringJoin["y", ToString[i]]],
     ToExpression[StringJoin["z", ToString[i]]]},
    {i, 1, NGP}];

(* Start filling the weight and coordinate tables *)

(* 1 Center point *)
coords[[1]] = {1 / 4, 1 / 4, 1 / 4};
weights[[1]] = 0.015066881743357949738327730990912;

(* 2..5: 4 first non-centers point near 0,0,0 and permutations *)
coords[[2]] = {0.057426917317356819579978725140823O,
   0.057426917317356819579978725140823O, 0.057426917317356819579978725140823O};
coords[[3]] = F1.coords[[2]] + {0, 0, 1};
coords[[4]] = F2.coords[[2]] + {0, 1, 0};
coords[[5]] = F3.coords[[2]] + {1, 0, 0};
weights[[2]] = 3.1866390464985314763201441565449 4 * 10 ^ (-3);
weights[[3]] = 3.1866390464985314763201441565449 4 * 10 ^ (-3);
weights[[4]] = 3.1866390464985314763201441565449 4 * 10 ^ (-3);
weights[[5]] = 3.1866390464985314763201441565449 4 * 10 ^ (-3);

(* 6..17: 12 points: first set of non-centers points near axis 0,
0,0-0.25,0.25,0.25 and permutations *)
coords[[6]] = {0.23129854365191466342385344099185 3,
   0.23129854365191466342385344099185 3, 0.48605102857060727870919871076850 7};
coords[[7]] = F1.coords[[6]] + {0, 0, 1};
coords[[8]] = F2.coords[[6]] + {0, 1, 0};
```

```
coords[[9]] = F3.coords[[6]] + {1, 0, 0};
coords[[10]] = {0.48605102857060727870919871076850 70,
   0.23129854365191466342385344099185 3, 0.23129854365191466342385344099185 3};
coords[[11]] = F1.coords[[10]] + {0, 0, 1};
coords[[12]] = F2.coords[[10]] + {0, 1, 0};
coords[[13]] = F3.coords[[10]] + {1, 0, 0};
coords[[14]] = {0.23129854365191466342385344099185 3,
   0.48605102857060727870919871076850 70, 0.23129854365191466342385344099185 3};
coords[[15]] = F1.coords[[14]] + {0, 0, 1};
coords[[16]] = F2.coords[[14]] + {0, 1, 0};
coords[[17]] = F3.coords[[14]] + {1, 0, 0};
weights[[6]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[7]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[8]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[9]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[10]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[11]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[12]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[13]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[14]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[15]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[16]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);
weights[[17]] = 7.2691564011109382427152201950077 7 * 10 ^ (-3);

(* 18..29: 12 points: second set of non-centers points near axis 0,
0,0-0.25,0.25,0.25 and permutations *)
coords[[18]] = {0.04756909881472295964602141920313 80,
   0.04756909881472295964602141920313 80, 0.60810798940152808607439022163335 0};
coords[[19]] = F1.coords[[18]] + {0, 0, 1};
coords[[20]] = F2.coords[[18]] + {0, 1, 0};
coords[[21]] = F3.coords[[18]] + {1, 0, 0};
coords[[22]] = {0.60810798940152808607439022163335 0,
   0.04756909881472295964602141920313 80, 0.04756909881472295964602141920313 80};
coords[[23]] = F1.coords[[22]] + {0, 0, 1};
coords[[24]] = F2.coords[[22]] + {0, 1, 0};
coords[[25]] = F3.coords[[22]] + {1, 0, 0};
coords[[26]] = {0.04756909881472295964602141920313 80,
   0.60810798940152808607439022163335 0, 0.04756909881472295964602141920313 80};
coords[[27]] = F1.coords[[26]] + {0, 0, 1};
coords[[28]] = F2.coords[[26]] + {0, 1, 0};
coords[[29]] = F3.coords[[26]] + {1, 0, 0};
weights[[18]] = 4.3019459936652776758729763917751 9 * 10 ^ (-3);
weights[[19]] = 4.3019459936652776758729763917751 9 * 10 ^ (-3);
weights[[20]] = 4.3019459936652776758729763917751 9 * 10 ^ (-3);
weights[[21]] = 4.3019459936652776758729763917751 9 * 10 ^ (-3);
```

```
weights[[22]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[23]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[24]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[25]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[26]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[27]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[28]] = 4.30194599366527767587297639177519 * 10^(-3);
weights[[29]] = 4.30194599366527767587297639177519 * 10^(-3);

(* Draw reference tetrahedron with integration points *)
Graphics3D[{
  Flatten[{
    Line[{{0, 0, 0}, {0, 0, 1}, {0, 1, 0},
      {1, 0, 0}, {0, 0, 1}, {0, 0, 0}, {0, 1, 0}, {0, 0, 0}, {1, 0, 0}}],
    Table[{Sphere[coords[[i]], (0.01 * Abs[weights[[i]]]) ^ (1/3)]}, {i, 1, NGP}]
   }]
 }]

(* Test whether accurate up to 6th order *)
Print["Accurate up to sixth order?"]
$Assumptions = {Element[{x, y, z}, Reals], x > 0, y > 0, z > 0};
Funk[x_, y_, z_] = (x + 0.001) ^1 * (y + 1.1) ^2 * (z + 0.1) ^2;
Print["Numerical integration by MMA minus numerical
    integration by IP-Scheme on 5th order polynomial: ", NumberForm[
   Integrate[Integrate[Integrate[Funk[x, y, z], {x, 0, 1 - y - z}], {y, 0, 1 - z}],
     {z, 0, 1}] - Sum[Funk[coords[[i, 1]], coords[[i, 2]], coords[[i, 3]]] *
      weights[[i]], {i, 1, NGP}], 20]];
Funk[x_, y_, z_] = (x + 0.001) ^2 * (y + 1.1) ^2 * (z + 0.1) ^2;
Print["Numerical integration by MMA minus numerical
    integration by IP-Scheme on 6th order polynomial: ", NumberForm[
   Integrate[Integrate[Integrate[Funk[x, y, z], {x, 0, 1 - y - z}], {y, 0, 1 - z}],
     {z, 0, 1}] - Sum[Funk[coords[[i, 1]], coords[[i, 2]], coords[[i, 3]]] *
      weights[[i]], {i, 1, NGP}], 20]];
Funk[x_, y_, z_] = (x + 0.001) ^2 * (y + 1.1) ^3 * (z + 0.1) ^2;
Print["Numerical integration by MMA minus numerical
    integration by IP-Scheme on 7th order polynomial: ", NumberForm[
   Integrate[Integrate[Integrate[Funk[x, y, z], {x, 0, 1 - y - z}], {y, 0, 1 - z}],
     {z, 0, 1}] - Sum[Funk[coords[[i, 1]], coords[[i, 2]], coords[[i, 3]]] *
      weights[[i]], {i, 1, NGP}], 20]];
Funk[x_, y_, z_] = (x + 0.001) ^2 * (y + 1.1) ^3 * (z + 0.1) ^3;
Print["Numerical integration by MMA minus numerical
    integration by IP-Scheme on 8th order polynomial: ", NumberForm[
   Integrate[Integrate[Integrate[Funk[x, y, z], {x, 0, 1 - y - z}], {y, 0, 1 - z}],
     {z, 0, 1}] - Sum[Funk[coords[[i, 1]], coords[[i, 2]], coords[[i, 3]]] *
      weights[[i]], {i, 1, NGP}], 20]];
```

```
Funk[x_, y_, z_] = (x + 0.001) ^3 * (y + 1.1) ^3 * (z + 0.1) ^3;
Print["Numerical integration by MMA minus numerical
    integration by IP-Scheme on 9th order polynomial: ", NumberForm[
   Integrate[Integrate[Integrate[Funk[x, y, z], {x, 0, 1 - y - z}], {y, 0, 1 - z}],
     {z, 0, 1}] - Sum[Funk[coords[[i, 1]], coords[[i, 2]], coords[[i, 3]]] *
      weights[[i]], {i, 1, NGP}], 20]];

(* Export weights and coordinates for use in UEL *)
str = OpenWrite["~/29GP.txt"];
WriteString[str,
  StringJoin["\n! Integration points and weights for reference element
     ((0 0 0) (1 0 0) (0 1 0) (0 0 1))\n"]];
WriteString[str, StringJoin["      double precision GP(29,4)\n"]];
For[i = 1, i ≤ 29, i++,
 WriteString[str, StringJoin["      GP(", ToString[i],
   ",1) =", ToString[FortranForm[coords[[i, 1]]]], "\n"]];
 WriteString[str, StringJoin["      GP(", ToString[i], ",2) =",
   ToString[FortranForm[coords[[i, 2]]]], "\n"]];
 WriteString[str, StringJoin["      GP(", ToString[i], ",3) =",
   ToString[FortranForm[coords[[i, 3]]]], "\n"]];
 WriteString[str, StringJoin["      GP(", ToString[i],
   ",4) =", ToString[FortranForm[weights[[i]]]], "\n"]];
]
Close[str]
```

Out[147]=



```
Accurate up to sixth order?

Numerical integration by MMA minus numerical integration
   by IP-Scheme on 5th order polynomial: -6.418476861114186 × 10^-17

Numerical integration by MMA minus numerical integration
   by IP-Scheme on 6th order polynomial: 2.732189474663471 × 10^-17

Numerical integration by MMA minus numerical integration
   by IP-Scheme on 7th order polynomial: -2.028334428368084 × 10^-9

Numerical integration by MMA minus numerical integration
   by IP-Scheme on 8th order polynomial: -2.279332739609299 × 10^-8

Numerical integration by MMA minus numerical integration
   by IP-Scheme on 9th order polynomial: -2.119993284594086 × 10^-7
```

Out[164]= /home/gluege/29GP.txt

In[167]:=
```
coords // MatrixForm
weights // MatrixForm
```

Out[167]//MatrixForm=

$\frac{1}{4}$	$\frac{1}{4}$	
0.0574269173173568195799787251408230	0.0574269173173568195799787251408230	0.05742
0.0574269173173568195799787251408230	0.0574269173173568195799787251408230	0.82771
0.0574269173173568195799787251408230	0.827719248047929541260063824577531	0.05742
0.827719248047929541260063824577531	0.0574269173173568195799787251408230	0.05742
0.231298543651914663423853440991850	0.231298543651914663423853440991850	0.4860
0.486051028570607278709198710768510	0.231298543651914663423853440991850	0.0513
0.486051028570607278709198710768510	0.051351884125563394443094407247790	0.2312
0.051351884125563394443094407247790	0.486051028570607278709198710768510	0.2312
0.4860510285706072787091987107685070	0.231298543651914663423853440991850	0.2312
0.231298543651914663423853440991850	0.231298543651914663423853440991850	0.0513
0.231298543651914663423853440991850	0.051351884125563394443094407247790	0.48605
0.051351884125563394443094407247790	0.231298543651914663423853440991850	0.2312
0.231298543651914663423853440991850	0.4860510285706072787091987107685070	0.2312
0.231298543651914663423853440991850	0.4860510285706072787091987107685070	0.0513
0.231298543651914663423853440991850	0.051351884125563394443094407247790	0.2312
0.051351884125563394443094407247790	0.231298543651914663423853440991850	0.48605
0.0475690988147229596460214192031380	0.0475690988147229596460214192031380	0.6081
0.608107989401528086074390221633350	0.0475690988147229596460214192031380	0.2967
0.608107989401528086074390221633350	0.296753812969025994633566939960370	0.04756
0.296753812969025994633566939960370	0.608107989401528086074390221633350	0.04756
0.608107989401528086074390221633350	0.0475690988147229596460214192031380	0.04756
0.0475690988147229596460214192031380	0.0475690988147229596460214192031380	0.2967
0.0475690988147229596460214192031380	0.296753812969025994633566939960370	0.6081
0.296753812969025994633566939960370	0.0475690988147229596460214192031380	0.04756
0.0475690988147229596460214192031380	0.608107989401528086074390221633350	0.04756
0.0475690988147229596460214192031380	0.608107989401528086074390221633350	0.2967
0.0475690988147229596460214192031380	0.296753812969025994633566939960370	0.04756
0.296753812969025994633566939960370	0.0475690988147229596460214192031380	0.6081

Out[168]//MatrixForm=

$$\begin{pmatrix} 0.01506688174335794973832773099909 1 \\ 0.003186639046498531476320144156544 94 \\ 0.003186639046498531476320144156544 94 \\ 0.003186639046498531476320144156544 94 \\ 0.003186639046498531476320144156544 94 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.007269156401110938242715220195007 8 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \\ 0.004301945993665277675872976391775 2 \end{pmatrix}$$