1 Ciarlet's elastic energy

The Ciarlet-Geymonat strain energy is close to the St.-Venant-Kirchhoff strain energy, but in contrast to the latter it is polyconvex in $\mathbf{F}$,

$$w(\mathbf{F}) = \frac{\lambda}{4}(I\!I\!I - \ln I\!I\!I) + \frac{\mu}{2}(I - \ln I\!I\!I). \tag{1}$$

It serves as starting point for our elastic strain gradient extension. The roman numbers I and $I\!I\!I$ denote the first and third principle invariants of the Cauchy-Green stretch tensors $\mathbf{B}$ and $\mathbf{C}$, λ and μ are Lamé's constants.

2 Strain gradient extension

To comply with the principle of invariance under rigid body motion, w is best formulated in terms of material deformation measures. While we use $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ for the first gradient, we need to define a similar material stretch measure for the second gradient $\mathbf{F}_3 = \mathbf{x} \otimes \nabla_0 \otimes \nabla_0$. A canonical extension is

$$\mathbf{C}_3 = \mathbf{F}^T\mathbf{F}_3. \tag{2}$$

A simple invariant is the norm, which we term g to indicate the strain gradient contribution,

$$g = \sqrt{\mathbf{C}_3 \cdot\cdot\cdot \mathbf{C}_3}. \tag{3}$$

We extend w by a summand that contains g. We choose

$$w_g(\mathbf{F}) = \frac{\lambda}{4}(I\!I\!I - \ln I\!I\!I) + \frac{\mu}{2}(I - \ln I\!I\!I) + \frac{\alpha}{2}g^2, \tag{4}$$

where α is a new material parameter.

3 Stress-Strain relation

We consider the first Piola-Kirchhoff stresses $\mathbf{T}$ as most practical for our purpose. Especially from an implementation point of view, we can write down the local balance of momentum by using the material nabla operator ∇_0,

$$\rho(\mathbf{a} - \mathbf{b}) = \mathbf{T} \cdot \nabla_0 - \mathbf{T}_3 \cdot\cdot (\nabla_0 \otimes \nabla_0). \tag{5}$$

Thus we need in the FE implementation the same operators ∇_0 and $\nabla_0 \otimes \nabla_0$ for calculating the first and second gradient and for calculating the residuals, which simplifies the implementation considerably. We refer to $\mathbf{T}_3$ thus as the first Piola-Kirchhoff stress tensor of third order, which is derived from w similar to the usual first Piola-Kirchhoff stresses,

$$\mathbf{T} = \frac{\partial w}{\partial \mathbf{F}}, \tag{6}$$

$$\mathbf{T}_3 = \frac{\partial w}{\partial \mathbf{F}_3}. \tag{7}$$

We summarize ρ_0 with the material constants, since it is also a constant. The derivatives evaluate to

$$\mathbf{T} = \mu\mathbf{F} + \left(\frac{\lambda}{2}(I\!I\!I - 1) - \mu\right)\mathbf{F}^{-T} + \alpha F_{mjl}C_{njl}\mathbf{e}_m \otimes \mathbf{e}_n, \tag{8}$$

$$\mathbf{T}_3 = \alpha\mathbf{F}\mathbf{C}_3 = \alpha\mathbf{B}\mathbf{F}_3. \tag{9}$$

The number of indices distinguishes $\mathbf{F}, \mathbf{F}_3$ and $\mathbf{C}, \mathbf{C}_3$ in the index notation. Additional to the stress-strain (gradient) relation, we need the linearisations,

$$\frac{\partial \mathbf{T}}{\partial \mathbf{F}} = \mu \mathbb{I} - \left(\frac{\lambda}{2}(I\!I\!I - 1) - \mu \right) (F^{-1}_{jk} F^{-T}_{il})_{ijkl} + \lambda I\!I\!I \mathbf{F}^{-T} \otimes \mathbf{F}^{-T} + \alpha (F_{ixy} F_{kxy} \delta_{jl})_{ijkl}, \tag{10}$$

$$\frac{\partial \mathbf{T}}{\partial \mathbf{F}_3} = \alpha (\delta_{ik} C_{jlm} + F_{kj} F_{ilm})_{ijklm}, \tag{11}$$

$$\frac{\partial \mathbf{T}_3}{\partial \mathbf{F}} = \alpha (\delta_{il} C_{mjk} + F_{im} F_{ljk})_{ijklm}, \tag{12}$$

$$\frac{\partial \mathbf{T}_3}{\partial \mathbf{F}_3} = \alpha \mathbf{B} \mathbb{I}^6 = \alpha (B_{il} \delta_{jm} \delta_{kn})_{ijklmn} \tag{13}$$